AD621869

MEMORANDUM
RM-4717-PR
SEPTEMBER 1965

ON ANALOGUES OF POINCARÉ-LYAPUNOV THEORY FOR MULTIPOINT BOUNDARY-VALUE PROBLEMS

Richard Bellman

PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

MEMORANDUM
RM-4717-PR
SEPTEMBER 1965

ON ANALOGUES OF POINCARÉ-LYAPUNOV THEORY FOR MULTIPOINT BOUNDARY-VALUE PROBLEMS

Richard Bellman

This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-700—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

The RAND Corporation
1700 MAIN ST • SANTA MONICA • CALIFORNIA • 90406

Published by The RAND Corporation

PREFACE

Control theory is studied because of its application to the guidance of mechanical, electronic, economic, and other processes. The solution of a number of its deterministic problems requires the minimization of a certain functional. This Memorandum considers the behavior of such solutions as the duration of control becomes infinite.

SUMMARY

The problem we shall treat here is connected with the minimization of

$$J(u) = \int_0^T (u'^2 + u^2 + 2G(u))dt,$$

over all scalar functions u subject to $u(0) = c$. The Euler equation is

$$u'' - u - g(u) = 0,$$

where $g = G'$, and u is subject to the free boundary condition $u'(T) = 0$. We wish to consider the question of existence and uniqueness of the solution and the determination of limiting behavior as $T \to \infty$. As might be expected, and is indeed the case, there are close interconnections with classical Poincaré—Lyapunov theory. We may consider investigations of the present type as part of an extensive generalization of that part of classical stability theory which concerns itself with initial-value problems.

CONTENTS

ON ANALOGUES OF POINCARÉ–LYAPUNOV THEORY FOR MULTIPOINT BOUNDARY-VALUE PROBLEMS

1. INTRODUCTION

A number of problems in deterministic control theory can be cast in the mold of a minimization of a functional of the form

$$(1.1) \qquad J(x,y) = \int_0^T g(x,y)dt,$$

over all functions $y(t)$ where x and y are related by means of the differential equation

$$(1.2) \qquad \frac{dx}{dt} = h(x,y), \quad x(0) = c.$$

(See [1].) The minimizing vectors will be functions of the initial value, c, the duration of the process, T, and, of course, t. It is of considerable interest from the viewpoints of both theory and application to study the behavior of the solution as $T \to \infty$. This investigation of asymptotic control theory was begun in a recent paper [2].

The problem we shall treat here is connected with the minimization of

$$(1.3) \qquad J(u) = \int_0^T (u'^2 + u^2 + 2G(u))dt,$$

over all scalar functions u subject to $u(0) = c$. The Euler equation is

$$u'' - u - g(u) = 0, \tag{1.4}$$

where $g = G'$, and u is subject to the free boundary condition $u'(T) = 0$. We wish to consider the question of existence and uniqueness of the solution and the determination of limiting behavior as $T \to \infty$. As might be expected, and is indeed the case, there are close interconnections with classical Poincaré-Lyapunov theory [3]. We may consider investigations of the present type as part of an extensive generalization of that part of classical stability theory which concerns itself with initial-value problems.

Similar problems arise in transport theory in connection with reflected, transmitted, and internal fluxes in plane-parallel slabs of finite thickness. It is important to demonstrate that the fluxes associated with the semi-infinite medium can be obtained as limits of the corresponding quantities and to determine the degree of convergence; see [4], [5], [6].

2. THE EQUATION $u'' - u = 0$

Let us begin with a derivation of the solution of

$$(2.1) \qquad u'' - u = 0, \quad u(0) = c, \quad u'(T) = 0.$$

It is easy to write down the explicit solution

$$(2.2) \qquad u(t) = c\,\frac{\cosh\,(t - T)}{\cosh T}.$$

To simplify the subsequent notation, let us write

$$(2.3) \qquad \begin{aligned} S(t) &= \sinh t, \\ C(t) &= \cosh t. \end{aligned}$$

Then,

$$(2.4) \qquad u(t) = c\,\frac{C(T - t)}{C(T)}.$$

As $T \to \infty$, we see that

$$(2.5) \qquad u(t) = c(e^{-t} + e^{t-2T})(1 + 0(e^{-T}))$$

for $0 \leq t \leq T$. In the limit, the solution is determined by a finiteness condition at $t = \infty$. We expect the same phenomenon to occur in the nonlinear case.

3. THE EQUATION $u'' - u = f$

We also need the solution of

$$(3.1) \qquad u'' - u = f, \quad u(0) = 0, \quad u'(T) = 0.$$

It is easy to verify that the solution has the form

$$(3.2) \qquad u(t) = c_1 S(t) + \int_0^t S(t - t_1) f(t_1) dt_1,$$

where c_1 is a parameter to be determined. Using the condition $u'(T) = 0$, we see that

$$(3.3) \qquad c_1 = -\int_0^T \frac{C(T - t_1) f(t_1) dt_1}{C(T)}.$$

Thus, after some collection of terms, we see that

$$(3.4) \qquad u = -\int_0^t \frac{C(T - t) S(t_1) f(t_1) dt_1}{C(T)} - \int_t^T \frac{C(T - t_1) S(t) f(t_1) dt_1}{C(T)} = \int_0^T K(t, t_1, T) f(t_1) dt_1.$$

The Green's function, $K(t, t_1, T)$, is nonpositive, is symmetric as it should be, and possesses a very simple limiting behavior. Namely, as $T \to \infty$,

$$K(t,t_1,T) \sim \frac{e^{-(t-t_1)}}{2}, \quad 0 \le t_1 \le t, \tag{3.5}$$

$$K(t,t_1,T) \sim \frac{e^{-(t_1-t)}}{2}, \quad t \le t_1 < \infty .$$

4. THE EQUATION $u'' - u = g(u)$

Let us now consider the equation

$$(4.1) \qquad u'' - u = g(u), \quad u(0) = c, \quad u'(T) = 0.$$

In order to establish the existence of a solution, we employ the classical method of converting the differential equation into an integral equation. Letting u_0 denote the solution of (2.1), we write, using the results of Sec. 3,

$$(4.2) \qquad u = u_0 + \int_0^t K(t,t_1,T)g(u(t_1))dt_1.$$

At this point, fixed-point techniques could be employed to obtain the existence of a solution; see [7]. Let us, however, as in [1], use successive approximations, writing

$$(4.3) \qquad u_{n+1} = u_0 + \int_0^t K(t,t_1,T)g(u_n(t_1))dt_1.$$

If we assume that $|c|$ is sufficiently small, that $g(u) = o(u)$ as $u \to 0$, and that $g(u)$ satisfies a Lipschitz condition for $|u| \leq b$, there is no difficulty in establishing the existence and uniqueness of the solution of (4.1).

A first step is a demonstration of the fact that $|u_n(t)| \leq 2|c|$ for $0 \leq t \leq T$, for $|c| \ll 1$. This is readily established inductively.

5. ASYMPTOTIC BEHAVIOR AS $T \to \infty$

Let us now turn to the question of asymptotic behavior as $T \to \infty$. Perhaps the simplest approach is to use the companion function obtained by means of a formal approach to the limit in (4.2),

$$v = v_0 + \int_0^t K(t,t_1) g(v(t_1)) dt_1. \tag{5.1}$$

Here

$$v_0(t) = \lim_{T\to\infty} u_0(t), \tag{5.2}$$

$$K(t,t_1) = \lim_{T\to\infty} K(t,t_1,T).$$

It is easy to verify that v is the solution of

$$v'' - v = g(v), \quad v(0) = c, \tag{5.3}$$

which approaches zero as $t \to \infty$. The existence and uniqueness of this solution is guaranteed by classical Poincaré–Lyapunov theory; see [1].

Let us then write

$$u = v + (u_0 - v) + \int_0^t K(t,t_1) g(u) dt_1 \tag{5.4}$$

$$+ \int_0^t (K(t,t_1,T) - K(t,t_1)) g(u) dt_1.$$

From this and the properties of u_0 and $K(t,t_1,T)$, it is easily seen that $\lim_{T\to\infty} u(t) = v(t)$.

6. DISCUSSION

It is not difficult to extend these results to vector-matrix systems. This will be carried out in subsequent papers. In addition, it is desirable to obtain an asymptotic expansion of $u(t)$ as $T \to \infty$. This will be done in several ways.

Another interesting question is that of the relation between the minimization of

$$(6.1) \qquad J_{\infty}(u) = \int_0^{\infty} (u'^2 + u^2 + 2G(u))dt,$$

and the minimization of $J(u)$ as given in (1.3). As the examples $G(u) = \pm u^4$ show, the situation is a good deal more complicated than that described above for the corresponding equations.

REFERENCES

1. Bellman, R., Adaptive Control Processes: A Guided Tour, Princeton University Press, Princeton, New Jersey, 1961.
2. Bellman, R., and R. Bucy, "Asymptotic Control Theory," J. SIAM Control, Vol. 2, 1964, pp. 11–18.
3. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill Book Company, Inc., New York, 1953.
4. Bellman, R., R. Kalaba, and G. M. Wing, "Invariant Imbedding and Mathematical Physics—I: Particle Processes," J. Math. Phys., Vol. 1, 1960, pp. 280–308.
5. Bellman, R., H. Kagiwada, R. Kalaba, and M. Prestrud, Invariant Imbedding and Time-dependent Processes, American Elsevier Publishing Company, Inc., New York, 1964.
6. Bellman, R., and R. Kalaba, Quasilinearization and Nonlinear Boundary-value Problems, American Elsevier Publishing Company, Inc., New York, 1965.
7. Bellman, R., "On the Boundedness of Solutions of Nonlinear Differential and Difference Equations," Trans. Amer. Math. Soc., Vol. 62, 1947, pp. 357–386.

DOCUMENT CONTROL DATA

1. ORIGINATING ACTIVITY THE RAND CORPORATION	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP
3. REPORT TITLE ON ANALOGUES OF POINCARE-LYAPUNOV THEORY FOR MULTIPOINT BOUNDARY-VALUE PROBLEMS	
4. AUTHOR(S) (Last name, first name, initial) Bellman, Richard	
5. REPORT DATE September 1965	6a. TOTAL NO. OF PAGES 18
	6b. NO. OF REFS. 7
7. CONTRACT or GRANT NO. AF 49(638)-700	8. ORIGINATOR'S REPORT NO. RM-4717-PR
9a. AVAILABILITY/LIMITATION NOTICES	9b. SPONSORING AGENCY United States Air Force Project RAND

10. ABSTRACT

A study of control theory and of its application to the guidance of mechanical, electronic, economic, and other processes. The solution of a number of its deterministic problems requires the minimization of a certain functional. The Memorandum considers the behavior of such solutions as the duration of control becomes infinite.

11. KEY WORDS

Control theory
Differential equations
Mathematics